BACKPAPER: ALGEBRA II

The total points is **100**.

(1) (10 points) Let K/F be a Galois extension of degree n. Show that there exist an irreducible polynomial $g(x)$ of degree n in $F[x]$ such that $g(x)$ splits into linear factors in $K[x]$.

(2) (10 points) Give an example of a finite normal field extension which is not a Galois extension.

(3) (15+15=30 points) Prove or disprove.
   (a) If F is a subfield of some cyclotomic field over $\mathbb{Q}$ then F/Q is a Galois extension with cyclic Galois group.
   (b) Every finite extension of a finite field is Galois.

(4) (15+15=30 points) Let F be a field. Let $f(x) = f_1(x)f_2(x) \in F[x]$ be a factorization into irreducible factors and assume $f(x)$ is separable. Let K be the splitting field of f. Let K_1 and K_2, contained in K, be the splitting fields of f_1 and f_2 respectively. Let G_1 and G_2 be the Galois group of K_1/F and K_2/F respectively. Assume that $f_2(x)$ is irreducible in $K_1[x]$. Show by an example that the Galois group of K/F need not be $G_1 \times G_2$. Let $\alpha \in K$ be a root of $f_2(x)$. If $K_2 = F(\alpha)$, show that Galois group of K/F is $G_1 \times G_2$.

(5) (10+10=20 points) Let a real number a be constructible by straight edge and compass. Show that the Galois group of the Galois closure of $\mathbb{Q}(a)/\mathbb{Q}$ is a solvable group.
Let $K/\mathbb{Q}$ be a Galois extension with Galois group A_5. Let $a \in K \setminus \mathbb{Q}$ be a real number, show that a is not constructible by straight edge and compass.